On Stability and Equilibria of Analog Feedback Networks

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August 12, 1994

Abstract

Both the analog Hopfield network [1] and the cellular neural network [2], [3] are special cases of the new M-lattice system, recently introduced to the signal processing community [4], [5], [6]. We prove that a subclass of the M-lattice is totally stable. This result also applies to the cellular neural network as a rigorous proof of its total stability. By analyzing the stability of fixed points, we derive the conditions for driving the equilibrium outputs of another subclass of the M-lattice to binary values. For the cellular neural network, this analysis is a precise formulation of an earlier argument based on circuit diagrams [2]. And for certain special cases of the analog Hopfield network, this analysis explains why the output variables converge to binary values even with non-zero neuron auto-connections. This behavior, observed in computer simulation by researchers for quite some time, is explained for the first time here.

1 Introduction

The neurally-inspired network, introduced by Hopfield in 1984 [1], and the cellular neural network, proposed by Chua *et al.* in 1988 [2], [7] have inspired tremendous world-wide research in both theory and practical applications. Both the Hopfield network and the cellular neural network are continuous-time continuous-valued (or analog) feedback (as opposed to feedforward) non-linear dynamical systems. Investigating their convergence properties and – in the cases of convergence – classifying the types of equilibrium output values have been the pursuit of many theoretical studies (see for example [1], [8], [9], [10], [11] for the Hopfield network and [2], [12], [3] for the cellular neural network). The binary optimization properties of the Hopfield network have been explored in many applications from the traveling salesman problem [13] to image processing [14], [15], while the cellular neural network has been used in more image processing and computational vision applications than there is room to cite here each separately [16].

The new M-lattice system [4], [5], [6] is rooted in the reaction-diffusion model, first proposed by Turing [17] to study *morphogenesis* and used later by others to explain the formation of animal patterns [18], [19]. But in contrast with the reaction-diffusion, the state variables of the M-lattice are guaranteed to be bounded. The M-lattice system is closely related to the analog Hopfield network and the cellular neural network, but has more flexibility in how its variables interact. Practical applications of the M-lattice have employed its pattern-forming, non-linear programming, and noise-shaping modes of operation [20].

The focus of this paper is on the total stability of the "diagonal-state clipped" M-lattice, a subclass of the M-lattice that contains the cellular neural network, and on the conditions that the system's parameters have to obey in order to assure binary outputs in steady state. The total stability is an important property because it guarantees convergence to equilibrium states. Binary outputs are often desired when solving coordinated-decisions problems, such as the traveling salesman problem, halftoning of images, *etc.*.

The present treatment proceeds in terms of the more general clipped M-lattice system, yielding new insight into the known behavior of the cellular neural network and establishing additional properties for the Hopfield network. Furthermore, the proofs of the total stability and of the binarization property presented here provide the cellular neural network community with more precise versions of the statements given in the original cellular neural network paper [2].

The rest of this document is organized as follows. Section 2 reviews several types of the M-lattice and discusses their relations to the Hopfield network and to the cellular neural network. Section 3 proves the total stability of the diagonal-state clipped M-lattice system and applies the result to the cellular neural network. Section 4 formulates the binarization constraint, proves its sufficiency, and then adapts it for the cellular neural network and for the Hopfield network. Finally, Section 5 summarizes the report.

2 Background: M-Lattice System

We now review the basics of the M-lattice [20]. The origin of the name "M-lattice system" comes from its roots in the reaction-diffusion paradigm, where M stands for the number of *morphogens*, or layers, in the lattice.

2.1 Notation

Let $\psi_i(t) \in \Re$ be a state variable as a function of time at each lattice point i, where $i = 1, \ldots, N$. Let $\chi_i(t)$ be an output variable, obtained from $\psi_i(t)$ via $\chi_i(t) = g(\psi_i(t))$. The "warping" function, $g(u)$, must be of a saturating type (*e.g.*, a sigmoid) so as to limit the range of the output variables. Construct $\vec{\psi}(t)$ and $\vec{\chi}(t)$ by concatenating $\psi_1(t), \ldots, \psi_N(t)$ and $\chi_1(t), \ldots, \chi_N(t)$, respectively into column vectors. In addition, let $\vec{\nabla}_{\vec{\psi}}$ be the gradient with respect to the elements of $\vec{\psi}$, $\vec{\nabla}_{\vec{\chi}}$ be the gradient with respect to the elements of $\vec{\chi}$, and $\mathbf{J}_{\vec{c}}(\vec{\psi})$ be the Jacobian of $\vec{c}(\vec{\psi}(t))$ at $\vec{\psi}$.

2.2 Fundamental M-Lattice Formulation

Definition 2.1 *Suppose that the given functions, $f_i(\vec{\chi}(t))$, are continuous, differentiable, and bounded. Let the matrix $\mathbf{A}$ be real with all eigenvalues having negative real parts: $\mathbf{A} \in \Re^{N \times N}$, $\mathbf{A} = [a_{ij}]$, and $\forall\ i\ \Re\left(\lambda_i\left[\mathbf{A}\right]\right) < 0$. Define $\vec{f}(\vec{\chi}(t))$ by concatenating $f_1(\vec{\chi}(t)), \ldots, f_N(\vec{\chi}(t))$ into* *a column vector. Then the M-lattice system* [1] *is a possibly non-linear autonomous dynamical system, described by the following equation:*

$$\frac{d\vec{\psi}(t)}{dt} = \mathbf{A}\vec{\psi}(t) + \vec{f}(\vec{\chi}(t)). \tag{1}$$

A crucial property of the M-lattice system is that its state variables are bounded.

Lemma 2.1 *The state vector, $\vec{\psi}(t)$, of the class of M-lattice systems, (1) with the previously stated assumptions, is eventually confined to a bounded region. In other words:*

$$\left.\begin{array}{l} \mathcal{B}_r \stackrel{\text{def}}{=} \left\{\vec{\xi} \in \Re^N,\ r \in \Re_+ :\ \left\|\vec{\xi}\right\| < r\right\}, \\ \forall\ \vec{\psi}(t_0)\ \exists\ r = R \in \Re_+ :\ \lim_{t\to\infty} \vec{\psi}(t) \in \overline{\mathcal{B}}_R, \end{array}\right\} \tag{2}$$

where $\overline{\mathcal{B}}_R$ is the closure of $\mathcal{B}_R$.

Proof: The boundedness of the functions, $f_i(\vec{\chi}(t))$, can be expressed as follows:

$$\exists\ \beta_1 \in \Re_+ :\ \left\|\vec{f}(\vec{\chi}(t))\right\| \ \leq\ \beta_1, \quad \text{where} \quad \left\|\vec{f}(\vec{\chi}(t))\right\| \ =\ \left(\left[\vec{f}(\vec{\chi}(t))\right]^T \left[\vec{f}(\vec{\chi}(t))\right]\right)^{\frac{1}{2}}. \tag{3}$$

Let λ_1 be the smallest eigenvalue of $-\mathbf{A}$ and consider the following globally positive-definite function:

$$W(t) \ \stackrel{\text{def}}{=}\ W(\vec{\psi}(t)) = \frac{1}{2}\left[\vec{\psi}(t)\right]^T \left[\vec{\psi}(t)\right]. \tag{4}$$

From (1), the time derivative of $W(t)$ is:

$$\begin{aligned} \frac{dW(t)}{dt} &= \left[\vec{\psi}(t)\right]^T \left[\frac{d\vec{\psi}(t)}{dt}\right] \\ &= \left[\vec{\psi}(t)\right]^T \left[\mathbf{A}\vec{\psi}(t) + \vec{f}(\vec{\chi}(t))\right] \\ &= \left[\vec{\psi}(t)\right]^T \mathbf{A}\left[\vec{\psi}(t)\right] + \left[\vec{\psi}(t)\right]^T \left[\vec{f}(\vec{\chi}(t))\right] \\ &\leq -\lambda_1 \left\|\vec{\psi}(t)\right\|^2 + \beta_1 \left\|\vec{\psi}(t)\right\| \\ &= \left\|\vec{\psi}(t)\right\| \left(-\lambda_1 \left\|\vec{\psi}(t)\right\| + \beta_1\right). \end{aligned} \tag{5}$$

From (5), $\frac{dW(t)}{dt} < 0$ if $\left\|\vec{\psi}(t)\right\| > \frac{\beta_1}{\lambda_1}$. Hence, all solution trajectories eventually enter into the closed ball, $\overline{\mathcal{B}}_R$ (in the $\vec{\psi}$-space) of radius, $R = \frac{\beta_1}{\lambda_1}$, centered at $\vec{0}$, and stay there. And since the warping functions are saturating non-linearities, there is a corresponding compact subset in the $\vec{\chi}$-space, to which all solution trajectories are eventually confined [21], [8], [10], [11]. ∎

[1] This is the definition adapted for the present paper. The general M-lattice system is defined in [20] and allows non-linear interactions among the state variables.

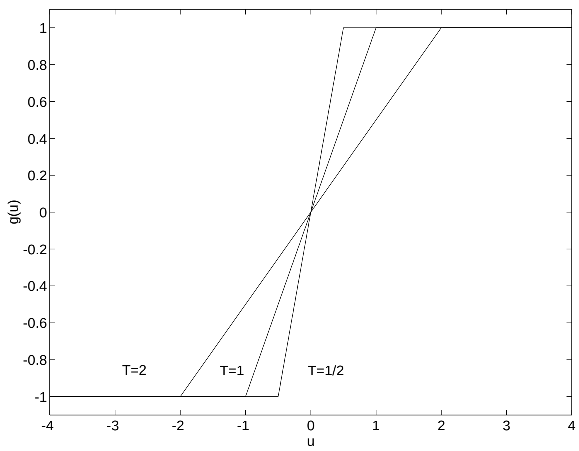

Figure 1: Plot of the clipping warping function for three different temperatures.

2.3 Clipped M-Lattice System

Consider the M-lattice system, (1), in which $\vec{f}(\vec{\chi}(t)) \stackrel{\text{def}}{=} \vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t))$ and $g(u)$ is the following "clipping" function:

$$\left.\begin{aligned} \frac{dg\,(u)}{du} &= \begin{cases} \dfrac{1}{T}, & |u| < T; \\ 0, & |u| \geq T. \end{cases} \\ g\,(u) &= \frac{1}{2}\left(\left|\frac{u}{T}+1\right| - \left|\frac{u}{T}-1\right|\right); \quad T \in \Re_+. \end{aligned}\right\} \tag{6}$$

This function is plotted in Figure 1.

Definition 2.2 *Suppose that the given function, $\Phi(\vec{\chi}(t))$, is continuous, twice-differentiable, and bounded. Let the matrix $\mathbf{A}$ be real, symmetric, and negative-definite: $\mathbf{A} \in \Re^{N\times N}$, $\mathbf{A} = [a_{ij}]$, $\mathbf{A} = \mathbf{A}^T$, and $\forall\ i\ \lambda_i\,[\mathbf{A}] < 0$. Then the clipped M-lattice system is the following possibly non-linear dynamical system:*

$$\frac{d\vec{\psi}(t)}{dt} = \mathbf{A}\vec{\psi}(t) + \vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t)). \tag{7}$$

We emphasize that the clipped M-lattice system is a special case of the M-lattice system. Notice that (7) differs from (1) in two aspects by employing particular choices of the warping function and of the non-linear interactions. Specifically, the warping function, used in (7), is (6), and $\vec{f}(\vec{\chi}(t))$ is a gradient, namely, $\vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t))$.

2.4 Comparison to Other Models

Several key aspects of the general M-lattice system, (1), with various saturating warping functions are unique when compared to other closely related models. Unlike the analog Hopfield network or the cellular neural network, the M-lattice system allows the $\mathbf{A}$ matrix to have off-diagonal elements. In addition, the interactions among the output variables, $\chi_i(t)$ can in principle be described by a very general non-linear function, $\vec{f}(\vec{\chi}(t))$. This flexibility enables the M-lattice system to capture the behavior of a wider variety of physical systems [20].

2.4.1 Comparison to Analog Hopfield Network

Suppose that the M-lattice is not clipped, but still restricts $\vec{f}(\vec{\chi}(t)) = \vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t))$. If, in addition, $\mathbf{A}$ is a diagonal matrix with negative elements on the main diagonal and $\Phi(\vec{\chi}(t))$ is a multilinear polynomial (*i.e.*, a polynomial whose independent variables have the powers zero or one [11]), then the M-lattice system with a sigmoidal warping function becomes the analog Hopfield network:

$$\frac{d\vec{\psi}(t)}{dt} = \mathbf{A}\vec{\psi}(t) + \vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t)), \tag{8}$$

$$\text{where } \mathbf{A} = \text{Diag } \{a_1, \ldots, a_N\}, \ \forall i \ a_i < 0; \tag{9}$$

$$\frac{\partial^2\Phi(\vec{\chi})}{\partial\chi_i^2} = 0 \ \forall i. \tag{10}$$

For binary outputs, the M-lattice system, similarly to the analog Hopfield network, is capable of optimization in the sense of the Hamming distance of one [11], [20].

2.4.2 Comparison to Cellular Neural Network

If $\mathbf{A}$ is a diagonal matrix with the same negative element on the main diagonal and $\vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t))$ is a circulant (or block-circulant) symmetric matrix, which represents the circular convolution with a linear shift-invariant FIR filter, then the clipped M-lattice system, (7), becomes the original cellular neural network [2], [7], [12], [3]:

$$\frac{d\vec{\psi}(t)}{dt} = \mathbf{A}\vec{\psi}(t) + \mathbf{B}\vec{\chi}(t) + \vec{s}, \text{ where} \tag{11}$$

$$\mathbf{A} = \text{Diag } \{a, \ldots, a\}, \ a < 0; \tag{12}$$

$$\mathbf{B} = \mathbf{H}^T\mathbf{H}, \text{ and} \tag{13}$$

$\mathbf{H}$ is a circulant matrix containing $h(\vec{n})$ (with $\vec{n} \in \mathcal{Z}^2$ for discrete 2-D space) in the first row.

$$\text{Alternatively: } \frac{d\psi(\vec{n},t)}{dt} = a(\vec{n}) * \psi(\vec{n},t) + b(\vec{n}) * \chi(\vec{n},t) + s(\vec{n}), \text{ where} \tag{14}$$

$$a(\vec{n}) = a\delta(\vec{n}) = -|a|\,\delta(\vec{n}); \tag{15}$$

$$b(\vec{n}) = h(-\vec{n}) * h(\vec{n}); \tag{16}$$

$$s(\vec{n}) \in [-1, 1]; \text{ and } \delta(\vec{n}) \text{ is the unit sample function in 2-D.} \tag{17}$$

A subclass of the clipped M-lattice system possesses total stability, which is manifested in asymptotic convergence, regardless of the initial conditions. This is the subject of the next section.

3 Proof of Total Stability

Totally stable dynamical systems are significant for engineering purposes. Not only are they guaranteed to converge to some fixed point, but they also provide mechanisms for guiding the trajectory of the system to the desired fixed point. Typically, this is accomplished by controlling the parameters of the system and specifying the initial conditions.

Definition 3.1 *An autonomous system is called totally stable if its every non-trivial trajectory approaches a finite asymptotically stable fixed point for any finite initial conditions.*

Note that Definition 3.1 does not imply that every fixed point of the given autonomous system has to be asymptotically stable, or even stable. The only requirement is that every solution trajectory must approach an equilibrium.

Proposition 3.1 *Consider a special case of the clipped M-lattice system, (7), in which* $\mathbf{A} = Diag\ \{a_1,\ \ldots,\ a_N\}, \forall\ i\ a_i < 0$. *Any solution trajectory of this diagonal-state clipped M-lattice system converges to a finite asymptotically stable fixed point, $\vec{\psi} \in \Re^N$ (or $\vec{\chi} \in [-1,1]^N$).*

Proof: The proof will be presented in steps:

1. Find a Lyapunov function, $E(t)$, and prove the stability of every fixed point.

2. Use the continuity of $E(t)$ to prove the asymptotic stability of every fixed point.

Figure 2 illustrates some of the main points of the proof.

3.1 Lyapunov Function and Stability

By Lemma 2.1, the $\psi_i(t)$ (state) variables and the $\chi_i(t)$ (output) variables of the clipped M-lattice system, (7), are bounded. Consider the following auxiliary function for the diagonal-state clipped M-lattice system:

$$E(t) \stackrel{\text{def}}{=} E(\vec{\chi}(t)) = \frac{T}{2}\vec{\chi}(t)^T \mathbf{A}\vec{\chi}(t) + \Phi(\vec{\chi}(t)). \tag{18}$$

Define:

$$\mathcal{I} = \{i\ :\ |\psi_i| < T\ (\text{or}\ |\chi_i| < 1)\} \tag{19}$$

and denote the complement of $\mathcal{I}$ by $\mathcal{I}^c$.

For $E(t)$ to fit the definition of a valid Lyapunov function near a stable fixed point, $\vec{\psi}$ (or $\vec{\chi}$), it is essential that:

$$\vec{\nabla}_{\vec{\psi}} E(\vec{\chi}(t)) = \vec{0} \tag{20}$$

at $\vec{\psi}$. From (18), $\vec{\nabla}_{\vec{\psi}} E(\vec{\chi}(t))$ is:

$$\begin{aligned}
\vec{\nabla}_{\vec{\psi}} E(\vec{\chi}(t)) &= \mathbf{J}_{\vec{\chi}}(\vec{\psi}(t))\vec{\nabla}_{\vec{\chi}} E(\vec{\chi}(t)) \\
&= \mathbf{J}_{\vec{\chi}}(\vec{\psi}(t))\left[T\mathbf{A}\vec{\chi}(t) + \vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t))\right] \\
&= \begin{cases} \left(\frac{1}{T}\right)\left(a_i\psi_i(t) + \frac{\partial\Phi(\vec{\chi}(t))}{\partial\chi_i(t)}\right), & i \in \mathcal{I} \\ 0, & i \in \mathcal{I}^c \end{cases} \\
&= \begin{cases} \left(\frac{1}{T}\right)\left(\frac{d\psi_i(t)}{dt}\right), & i \in \mathcal{I} \\ 0, & i \in \mathcal{I}^c. \end{cases}
\end{aligned} \tag{21}$$

Since both parts of the last equation in (21) identically vanish at a fixed point, the necessary condition in (20) is satisfied.

By assumption on the elements of $\mathbf{A}$, Definition 2.2, and (6), $E(t)$ is bounded above. In order to show that $E(t)$ is non-decreasing, we observe that according to (6), $\mathbf{J}_{\vec{\chi}}(\vec{\psi}(t))$ is a positive-semidefinite matrix. Then differentiating $E(t)$ with respect to time gives:

$$\begin{aligned} \frac{dE(t)}{dt} &= \left[\vec{\nabla}_{\vec{\psi}} E(\vec{\chi}(t))\right]^T \left[\frac{d\vec{\psi}(t)}{dt}\right] \\ &= \sum_i \left[\left\{ \begin{array}{ll} \left(\frac{1}{T}\right)\left(\frac{d\psi_i(t)}{dt}\right), & i \in \mathcal{I} \\ 0, & i \in \mathcal{I}^c \end{array} \right\} \left(\frac{d\psi_i(t)}{dt}\right)\right] \\ &= \frac{1}{T}\sum_{i\in\mathcal{I}} \left[\frac{d\psi_i(t)}{dt}\right]^2 \quad = \quad T\sum_{i\in\mathcal{I}} \left[\frac{d\chi_i(t)}{dt}\right]^2 \quad \geq \quad 0. \end{aligned} \tag{22}$$

Thus, the auxiliary function, $E(t)$, is bounded above and non-decreasing, and its extrema coincide with the fixed points of the system. By this reasoning, one of the fixed points must correspond to the upper bound or the global maximum of $E(\vec{\psi}(t))$. Other fixed points correspond to the local maxima, the saddles, and the local minima of $E(\vec{\psi}(t))$. The saddles and the local minima of $E(\vec{\psi}(t))$ correspond to unstable fixed points, because $E(t)$ is non-decreasing along the non-trivial solution trajectories. For the same reason, the local maxima of $E(\vec{\psi}(t))$ correspond to stable fixed points, and since $E(\vec{\psi}(t))$ has a global maximum, the system has at least one stable fixed point. Hence, every trajectory of the diagonal-state clipped M-lattice system, (7), contains a stable fixed point, $\vec{\psi}$ (or $\vec{\chi}$), and $E(t)$ is a Lyapunov function.

3.2 Strengthening Stability to Asymptotic Stability

We now proceed to show that every such stable fixed point, $\vec{\psi}$ (or $\vec{\chi}$), is asymptotically stable as well. It is evident from (22) that:

$$\lim_{t\to\infty} \frac{dE(t)}{dt} = 0 \iff \lim_{t\to\infty} \frac{d\psi_i(t)}{dt} = \lim_{t\to\infty} \frac{d\chi_i(t)}{dt} = 0,\ \forall i \in \mathcal{I}. \tag{23}$$

Observe that since:

$$\frac{dg(\psi_i(t))}{d\psi_i(t)} \quad = \quad 0,\ \forall i \in \mathcal{I}^c, \tag{24}$$

$\lim\limits_{t\to\infty} \frac{dE(t)}{dt}$ is unaffected by $\lim\limits_{t\to\infty} \frac{d\psi_i(t)}{dt}$, $i \in \mathcal{I}^c$ (or $\lim\limits_{t\to\infty} \frac{d\chi_i(t)}{dt}$, $i \in \mathcal{I}^c$). In other words, once all $\psi_i(t)$, $i \in \mathcal{I}$ stop changing, $E(t)$ reaches a constant value and stays there forever, regardless of any possible dynamics of $\chi_i(t) \in \{-1, 1\}$, $i \in \mathcal{I}^c$.

The next step is to show that, in fact, no dynamics of $\chi_i(t) \in \{-1, 1\}$, $i \in \mathcal{I}^c$ are possible in the limit as $t \to \infty$. Formally, we must prove that:

$$\lim_{t\to\infty} \chi_i(t) = \chi_i,\ i \in \mathcal{I}^c \quad \text{and hence} \quad \lim_{t\to\infty} \vec{\chi}(t) = \vec{\chi} \iff \lim_{t\to\infty} \frac{d\vec{\chi}(t)}{dt} = \vec{0}. \tag{25}$$

The latter is true if the trajectory of $\vec{\chi}(t)$ reaches a level set of $E(t)$ and does not move any more (not even on the level set). Consider the following alternative expression for $\dfrac{dE(t)}{dt}$ in the limit as $t \to \infty$:

$$\lim_{t\to\infty} \frac{dE(t)}{dt} \quad = \quad \lim_{t\to\infty} \left\{ \left[\vec{\nabla}_{\vec{\chi}} E(\vec{\chi}(t))\right]^T \left[\frac{d\vec{\chi}(t)}{dt}\right] \right\} \quad = \quad 0, \tag{26}$$

where the last equality is due to (22) and (23). According to (26), $\vec{\nabla}_{\vec{\chi}} E(\vec{\chi}(t))$ tends to become orthogonal to the system's trajectory, $\vec{\chi}(t)$, in the limit as $t \to \infty$. This is consistent with (22) and means that the trajectory tends toward a level set of $E(t)$. Now, assume that the converse of the second statement in (25) is true. Then in the limit as $t \to \infty$, the trajectory traverses the level set. Thus, taking (23) into account, we have:

$$\exists i \in \mathcal{I}^c,\ \exists \tau \in \Re,\ \tau \to \infty :\ \left.\frac{d\chi_i(t)}{dt}\right|_{t\,=\,\tau} \quad \neq \quad 0. \tag{27}$$

Since $\chi_i(t) \in \{-1, 1\}$, $i \in \mathcal{I}^c$, the statement in (27) is equivalent to:

$$\exists i \in \mathcal{I}^c,\ \exists \tau \in \Re_+,\ \tau \to \infty,\ \exists \varepsilon \in \Re_+,\ \varepsilon \to 0 :\ \chi_i(\tau + \varepsilon) \quad = \quad -\chi_i(\tau). \tag{28}$$

In order to satisfy (28), the trajectory, $\vec{\chi}(t)$ or $\vec{\psi}(t)$, of the diagonal-state clipped M-lattice system along the sought level set of $E(t)$ must encounter a discontinuity. But this would violate the continuity of $E(t)$, thereby contradicting Definition 2.2. In other words, $\chi_i(t)$ cannot toggle without passing through the continuum of intermediate values, which is impossible in the limit as $t \to \infty$, for $E(\vec{\psi}(t))$ reaches a local maximum, and $\mathcal{I}$ and $\mathcal{I}^c$ become constant sets (*i.e.*, the counts of indices comprising the sets and the indices themselves stop changing with time). Hence, the conjecture in (25) follows.

Effectively, we have shown that the trajectory of the $\chi_i(t)$ (output) variables does not contain cycles. Then (23) becomes:

$$\lim_{t\to\infty} \frac{dE(t)}{dt} = 0 \iff \lim_{t\to\infty} \frac{d\psi_i(t)}{dt} = 0,\ \forall i \in \mathcal{I};\ \lim_{t\to\infty} \frac{d\vec{\chi}(t)}{dt} = \vec{0}. \tag{29}$$

Finally, we have to show that the $\psi_i(t)$ (state) variables converge as well. Note that since the output variables do not exhibit cycles, the functions, $f_i(\vec{\chi}(t))$, saturate with time. In other words:

$$\lim_{t\to\infty} \vec{f}(\vec{\chi}(t)) \quad = \quad \vec{F}, \tag{30}$$

where $\vec{F}$ is a constant vector. Using (30), we rewrite (7) in the limit as $t \to \infty$:

$$\lim_{t\to\infty} \frac{d\vec{\psi}(t)}{dt} \quad = \quad \lim_{t\to\infty} \mathbf{A}\vec{\psi}(t) + \vec{F}. \tag{31}$$

We recognize (31) as a driven linear system for large t with the constants, F_i, making up the driving terms. By the hypothesis, all eigenvalues of $\mathbf{A}$ are negative. Hence, every fixed point of

(31) is asymptotically stable, so that (31) converges to a constant set of states. In other words, $\lim_{t\to\infty} \frac{d\psi_i(t)}{dt} = 0$ (or $\lim_{t\to\infty} \frac{d\vec{\psi}(t)}{dt} = \vec{0}$).

As a result, (23) and (29) are strengthened to:

$$\lim_{t\to\infty} \frac{dE(t)}{dt} = 0 \iff \lim_{t\to\infty} \frac{d\vec{\psi}(t)}{dt} = \lim_{t\to\infty} \frac{d\vec{\chi}(t)}{dt} = \vec{0}. \tag{32}$$

This completes the proof. ∎

Remark 3.1 *If $T = 1$, $\vec{f}(\vec{\chi}(t)) \stackrel{\text{def}}{=} \vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t))$ is linear in $\chi_i(t)$, $\mathbf{J}_{\vec{f}}(\vec{\psi})$ is a circulant (or block-circulant) and symmetric matrix, and $a_i = -|a|$, $\forall\ i$, then the diagonal-state clipped M-lattice system reduces to the reciprocal cellular neural network [2], [16]:*

$$\frac{d\psi(\vec{n},t)}{dt} = -|a|\,\psi(\vec{n},t) + b(\vec{n}) * \chi(\vec{n},t) + s(\vec{n}) \quad or \tag{33}$$

$$\frac{d\psi_i(t)}{dt} = -|a|\,\psi_i(t) + \sum_j b_{i,j}\chi_j(t) + s_i. \tag{34}$$

Since the clipped M-lattice system, (7), permits arbitrary linear interactions among the state variables, non-linear interactions among the output variables, and allows any $T \in \Re_+$, it is a generalization of the diagonal-state clipped M-lattice system and of the reciprocal cellular neural network. However, the clipped M-lattice system is not totally stable for all parameter values [20].

4 Designing Clipped M-Lattice to Have Binary Outputs

As was mentioned in Section 1, many applications that rely on solving the so-called "co-ordinated decisions problems" require binary outputs. In this section, we derive conditions for driving the outputs of two subclasses of the M-lattice system to the corners of the N-dimensional hypercube, containing the space of the output variables.

4.1 General Condition for Binary Outputs

For notational convenience, denote $\frac{d\vec{\psi}(t)}{dt}$ in (7) by $\vec{c}(\vec{\psi}(t))$. Then the following lemma contains the goal of a design procedure that guarantees binary outputs.

Lemma 4.1 *Suppose that the clipped M-lattice system, (7), is totally stable and consider its evolution towards an asymptotically stable fixed point, $\vec{\psi}$. If $\mathbf{J}_{\vec{c}}(\vec{\psi})$ is such that $\exists i\ Re\left(\lambda_i\left[\mathbf{J}_{\vec{c}}(\vec{\psi})\right]\right) > 0$, then $\mathcal{I} = \emptyset$. In other words, $|\mathcal{I}| = 0$, $|\mathcal{I}^c| = N$ (*i.e., *all N state variables have $|\psi_i(t)| \geq T$), and the stable fixed point is $\vec{\chi} \in \{-1,1\}^N$.*

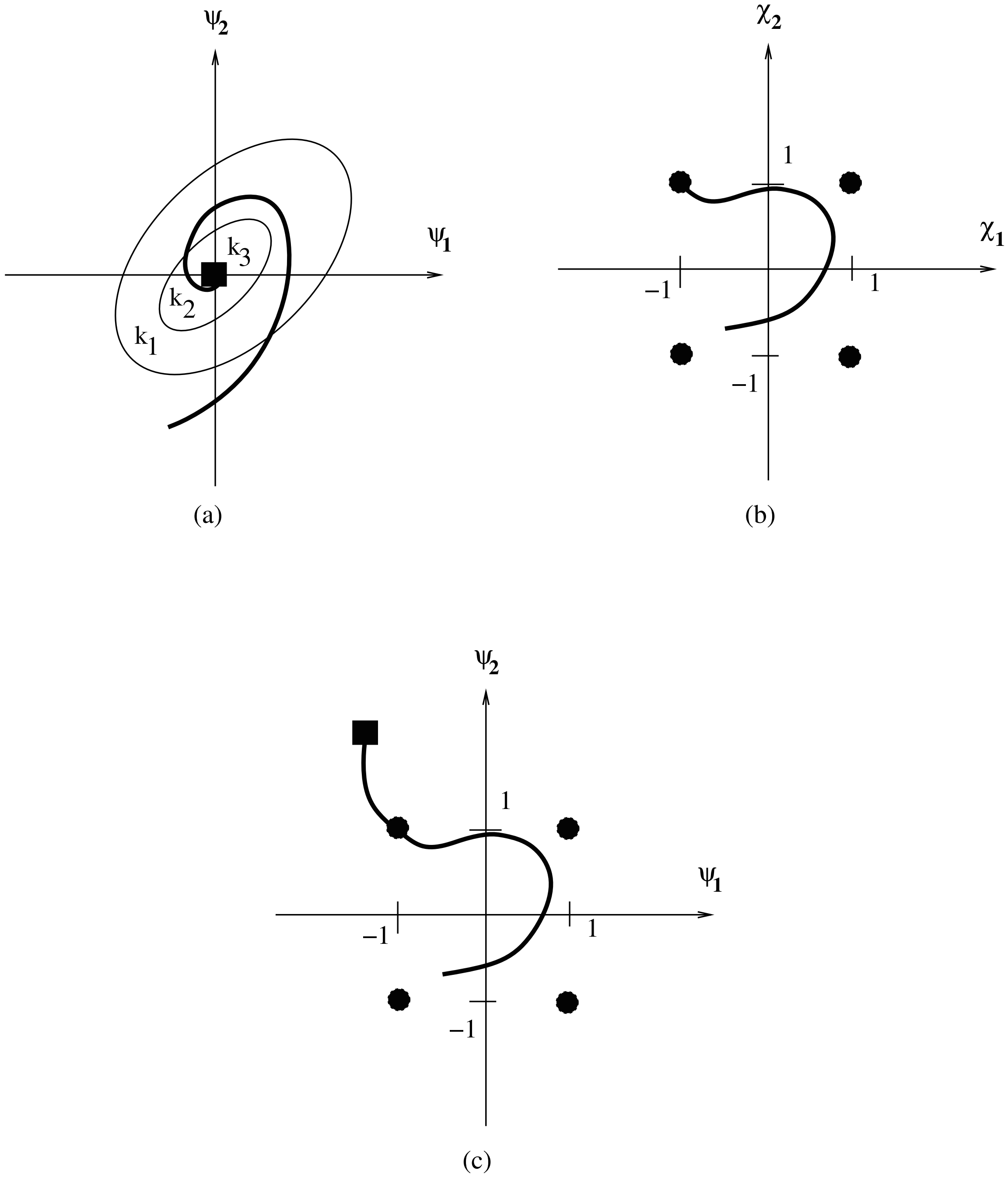

Figure 2: Illustrating several key ideas for the proof of the total stability of the diagonal-state clipped M-lattice system (here, $N = 2$, $T = 1$). The fixed points $\vec{\psi}$ are marked by squares, and the fixed points $\vec{\chi}$ are marked by disks. (a) When $1, 2 \in \mathcal{I}$, $|\psi_i| < T$. Then the trajectory of the state variables moves in the direction of increasing $E(t)$ till convergence. The level sets of $E(t)$ are marked by k_1, k_2, and k_3, the values of $E(t)$ at those contours. Here, $k_1 < k_2 < k_3$. (b) When $1, 2 \in \mathcal{I}^c$, $|\psi_i| \geq T$ and $\chi_i \in \{-1, 1\}$. Then the trajectory of the output variables goes to one of the fixed corner points, and stays there. (c) When $1, 2 \in \mathcal{I}^c$ and $\vec{\chi}(t)$ is already at a fixed point, $\vec{\psi}(t)$ may not have settled yet. In this case, $\vec{\psi}(t) \to \vec{\psi}$ in the limit as $t \to \infty$, where $\vec{\psi}$ is a stable node with $|\psi_i| \geq T$.

Proof: By Definition 2.2 and the continuity of $\vec{c}(\vec{\psi}(t))$, there exists a sufficiently small local neighborhood around the fixed point, for which (7) becomes [22], [23], [24]:

$$\begin{aligned} \frac{d\vec{\psi}(t)}{dt} &= \vec{c}(\vec{\psi}(t)) = \mathbf{J}_{\vec{c}}(\vec{\psi})(\vec{\psi}(t) - \vec{\psi}) \\ &= \left(\mathbf{A} + \mathbf{H}_{\Phi}(\vec{\psi})\mathbf{J}_{\vec{\chi}}(\vec{\psi})\right)(\vec{\psi}(t) - \vec{\psi}), \end{aligned} \tag{35}$$

where $\mathbf{H}_{\Phi}(\vec{\psi})$ is the Hessian of $\Phi(\vec{\chi}(t))$ at the fixed point, $\vec{\psi}(t) = \vec{\psi}$:

$$\mathbf{H}_{\Phi}(\vec{\psi}) \in \Re^{N\times N},\ \mathbf{H}_{\Phi}(\vec{\psi}) = \left[h_{ij}(\vec{\psi})\right],\ h_{ij}(\vec{\psi}) = \left.\frac{\partial^2\Phi(\vec{\chi}(t))}{\partial\chi_i(t)\partial\chi_j(t)}\right|_{\vec{\psi}(t) = \vec{\psi}}. \tag{36}$$

Assume that the premise of the total stability holds, but $|\mathcal{I}| > 0$ (*i.e.*, $\mathcal{I} \neq \emptyset$). According to the hypothesis on the eigenvalues of the Jacobian, the linearized (small-signal) system, (35), is unstable [25], [26]. Because of this, the overall (large-signal) clipped M-lattice system is also unstable [23], [24]. But this contradicts the assumption of total stability. Hence, the conclusion follows. ∎

Remark 4.1 *If $|\mathcal{I}| = 0$ for the clipped M-lattice system, (7), then $\mathbf{H}_{\Phi}(\vec{\psi})\mathbf{J}_{\vec{\chi}}(\vec{\psi}) = \mathbf{0}$, implying that every fixed point of the form, $\vec{\chi} \in \{-1, 1\}^N$, is asymptotically stable [24], [20].*

4.2 Solution for Circulant Symmetric Matrices

Now, consider a class of the clipped M-lattice system, in which $\mathbf{A}$ and $\mathbf{H}_{\Phi}(\vec{\psi})$ are restricted to be circulant. Such matrices correspond to 1-D finite impulse response (FIR) filters. For 2-D filters, $\mathbf{A}$ and $\mathbf{H}_{\Phi}(\vec{\psi})$ become block-circulant. In a circulant or a block-circulant matrix, the same number appears on the main diagonal. A 1-D filter, $h(n)$, is symmetric if $h(n) = h(N-n)$. For 2-D FIR filters, the symmetry requirement is: $h(n_x, n_y) = h(N - n_x,\ N - n_y)$. If the filter is symmetric, then the corresponding circulant or block-circulant matrix is also symmetric.

For example, circularly convolving with the following symmetric 1-D FIR filter:

$$h(n) = \left\{ \begin{array}{ll} h_0, & n = 0; \\ h_1, & n = 1; \\ h_1, & n = 2 \end{array} \right\} \tag{37}$$

is equivalent to multiplying on the left by the following symmetric circulant matrix:

$$H = \begin{bmatrix} h_0 & h_1 & h_1 \\ h_1 & h_0 & h_1 \\ h_1 & h_1 & h_0 \end{bmatrix}. \tag{38}$$

Analogously, if the symmetric 2-D FIR filter is:

$$h(n_x, n_y) = \left\{ \begin{array}{ll} h_{00}, & (n_x, n_y) = (0,0); \\ h_{01}, & (n_x, n_y) = (0,1); \\ h_{01}, & (n_x, n_y) = (0,2); \\ h_{10}, & (n_x, n_y) = (1,0); \\ h_{11}, & (n_x, n_y) = (1,1); \\ h_{12}, & (n_x, n_y) = (1,2); \\ h_{10}, & (n_x, n_y) = (2,0); \\ h_{12}, & (n_x, n_y) = (2,1); \\ h_{11}, & (n_x, n_y) = (2,2) \end{array} \right\}, \tag{39}$$

then the corresponding symmetric block-circulant matrix is:

$$H = \left[\begin{array}{ccc|ccc|ccc} h_{00} & h_{01} & h_{01} & h_{10} & h_{11} & h_{12} & h_{10} & h_{12} & h_{11} \\ h_{01} & h_{00} & h_{01} & h_{12} & h_{10} & h_{11} & h_{11} & h_{10} & h_{12} \\ h_{01} & h_{01} & h_{00} & h_{11} & h_{12} & h_{10} & h_{12} & h_{11} & h_{10} \\ \hline h_{10} & h_{12} & h_{11} & h_{00} & h_{01} & h_{01} & h_{10} & h_{11} & h_{12} \\ h_{11} & h_{10} & h_{12} & h_{01} & h_{00} & h_{01} & h_{12} & h_{10} & h_{11} \\ h_{12} & h_{11} & h_{10} & h_{01} & h_{01} & h_{00} & h_{11} & h_{12} & h_{10} \\ \hline h_{10} & h_{11} & h_{12} & h_{10} & h_{12} & h_{11} & h_{00} & h_{01} & h_{01} \\ h_{12} & h_{10} & h_{11} & h_{11} & h_{10} & h_{12} & h_{01} & h_{00} & h_{01} \\ h_{11} & h_{12} & h_{10} & h_{12} & h_{11} & h_{10} & h_{01} & h_{01} & h_{00} \end{array} \right]. \tag{40}$$

Using symmetric FIR filters as parameters of clipped M-lattice systems is convenient. First, the FIR computation is local in nature. Locality is beneficial, because it minimizes the inter-processor wiring requirement in VLSI realization and the amount of floating point operations in computer simulation. Second, the added symmetry makes the computation not only local, but also modular. Symmetric FIR hardware is a spatially-scalable structure, built out of identical computation / communication modules.

Moreover, as we will show, properly choosing just one parameter, $h(\vec{n} = \vec{0})$, is sufficient in order to drive the outputs of the diagonal-state clipped M-lattice system to ± 1. For notational simplicity, we have dropped the dependence of $\mathbf{H}_\Phi(\vec{\psi})$ on $\vec{\psi}$, but it should be clear that the requirements to be derived must hold for any value of $\vec{\psi}$. Simplifying the notation of $\mathbf{H}_\Phi(\vec{\psi})$ to $\mathbf{H}$ is strictly legitimate only when $\mathbf{H}_\Phi(\vec{\psi})$ is a constant matrix.

Thus, the goal is to satisfy the condition of Lemma 4.1 by choosing $h(\vec{0})$ such that $\mathbf{J}_{\vec{c}}(\vec{\psi})$ has at least one eigenvalue with a positive real part for $|\mathcal{I}| > 0$.

Proposition 4.1 *Assume that the clipped M-lattice system, (7), is totally stable. Suppose that* $\mathbf{A}$ *and* $\mathbf{H}$ *used in the clipped M-lattice system, (7), correspond to symmetric filters, and the following conditions are satisfied:*

$$\left\{\begin{array}{rcl} a(\vec{0})+\dfrac{h(\vec{0})}{T} - \displaystyle\sum_{\vec{n}\neq\vec{0}}|a(\vec{n})| & > & a(\vec{0})+\left|a(\vec{k})+\dfrac{h(\vec{k})}{T}\right| + \displaystyle\sum_{\vec{n}\neq\vec{0},\vec{n}\neq\vec{k}}|a(\vec{n})|, \\ a(\vec{0})+\dfrac{h(\vec{0})}{T} - \displaystyle\sum_{\vec{n}\neq\vec{0}}|a(\vec{n})| & > & 0, \\ \textit{where } \vec{k} & = & \arg\max\limits_{\vec{n}\neq\vec{0}}\left|a(\vec{n})+\dfrac{h(\vec{n})}{T}\right|. \end{array}\right. \tag{41}$$

Then $\vec{\chi}\in\{-1,1\}^N$.

Proof: Introduce additional notation:

$$\mathbf{B} \;=\; \mathbf{J}_{\vec{c}}(\vec{\psi}) \;=\; \left(\mathbf{A}+\mathbf{H}\mathbf{J}_{\vec{\chi}}(\vec{\psi})\right). \tag{42}$$

For the purposes of illustration, we will use the 3×3 example of (37) and (38):

$$\mathbf{B} \;=\; \begin{bmatrix} a_0+h_0\dfrac{dg(\psi_1)}{d\psi_1} & a_1+h_1\dfrac{dg(\psi_2)}{d\psi_2} & a_1+h_1\dfrac{dg(\psi_3)}{d\psi_3} \\ a_1+h_1\dfrac{dg(\psi_1)}{d\psi_1} & a_0+h_0\dfrac{dg(\psi_2)}{d\psi_2} & a_1+h_1\dfrac{dg(\psi_3)}{d\psi_3} \\ a_1+h_1\dfrac{dg(\psi_1)}{d\psi_1} & a_1+h_1\dfrac{dg(\psi_2)}{d\psi_2} & a_0+h_0\dfrac{dg(\psi_3)}{d\psi_3} \end{bmatrix}. \tag{43}$$

There is no loss of generality, and the proof is valid for $\mathbf{B}$ of any size.

Observe that the worst-case situation occurs when only one output variable has not yet saturated to ± 1 (*i.e.*, the trajectory of the clipped M-lattice system is moving along an edge of the N-dimensional hypercube):

$$\mathbf{B}_{\text{edge}} \;=\; \begin{bmatrix} a_0+\dfrac{h_0}{T} & a_1 & a_1 \\ a_1+\dfrac{h_1}{T} & a_0 & a_1 \\ a_1+\dfrac{h_1}{T} & a_1 & a_0 \end{bmatrix}. \tag{44}$$

The chosen value of h_0 must guarantee that the trajectory will not terminate on the edge, but head to a corner.

In order to see this, consider (43) at an interior point:

$$\mathbf{B}_{\text{int}} \;=\; \begin{bmatrix} a_0+\dfrac{h_0}{T} & a_1+\dfrac{h_1}{T} & a_1+\dfrac{h_1}{T} \\ a_1+\dfrac{h_1}{T} & a_0+\dfrac{h_0}{T} & a_1+\dfrac{h_1}{T} \\ a_1+\dfrac{h_1}{T} & a_1+\dfrac{h_1}{T} & a_0+\dfrac{h_0}{T} \end{bmatrix}. \tag{45}$$

Since $\mathbf{B}_{\text{int}}$ is circulant, it has the same value on the main diagonal. Recall that $\mathbf{A}$ is a circulant and negative-definite matrix, which implies that $a_0 < 0$. Then setting $h_0 > -Ta_0$ gives trace $[\mathbf{B}_{\text{int}}] > 0$. Since $\mathbf{B}_{\text{int}}$ is symmetric with positive trace, it possesses at least one positive eigenvalue. This makes interior fixed points unstable, and some output variable must saturate to ± 1.

With one output variable saturated, the candidate fixed points are on the faces of the hypercube, and:

$$\mathbf{B}_{\text{face}} = \begin{bmatrix} a_0 + \frac{h_0}{T} & a_1 + \frac{h_1}{T} & a_1 \\ a_1 + \frac{h_1}{T} & a_0 + \frac{h_0}{T} & a_1 \\ a_1 + \frac{h_1}{T} & a_1 + \frac{h_1}{T} & a_0 \end{bmatrix}. \tag{46}$$

Note that $\mathbf{B}_{\text{face}}$ is neither circulant nor symmetric, and one diagonal element is always negative. Hence, in order for $\mathbf{B}_{\text{face}}$ to have an eigenvalue with a positive real part, h_0 cannot be less than what it was in the case of $\mathbf{B}_{\text{int}}$. Therefore, since eigenvalues are continuous functions of matrix elements, $\mathbf{B}_{\text{edge}}$ depicts the worst-case scenario.

Also observe that all the $\mathbf{B}$ matrices, representing the edges of the hypercube, have the same eigenvalues. This is a consequence of the fact that the $\mathbf{B}$ matrix that describes an edge can be obtained from the $\mathbf{B}$ matrix that describes any other edge by performing one row exchange and one column exchange. When the total number of row and column exchanges is even, the characteristic polynomial and, consequently, the eigenvalues of a matrix are preserved. Since all the edges of the hypercube are equivalent in this sense, any convenient choice of $\mathbf{B}_{\text{edge}}$, such as the one in (44), is appropriate.

The remainder of the proof employs Gerschgorin's circle theorems [27], [25], [28], [26]. One Gerschgorin circle of $\mathbf{B}_{\text{edge}}$ is centered at $a_0 + \frac{h_0}{T}$, with the radius $r_0 = \sum_{\vec{n} \neq \vec{0}} |a(\vec{n})|$. All the other Gerschgorin circles of $\mathbf{B}_{\text{edge}}$ are centered at a_0 with the maximum radius $r_{\text{sup}} = \left| a(\vec{k}) + \frac{h(\vec{k})}{T} \right| + \sum_{\vec{n} \neq \vec{0}, \vec{n} \neq \vec{k}} |a(\vec{n})|$. It is known that an isolated Gerschgorin circle contains exactly one eigenvalue [27], [28]. According to the hypothesis, (41), the circle centered at $a_0 + \frac{h_0}{T}$ is isolated from the rest and is situated entirely in the right half plane. Hence, $\exists i \ \mathrm{Re}\left(\lambda_i \left[\mathbf{B}_{\text{edge}}\right]\right) > 0$. Then by Lemma 4.1, edges of the hypercube do not contain stable fixed points. Therefore, $\vec{\chi} \in \{-1, 1\}^N$. ∎

Remark 4.2 *The* $\mathbf{B}_{\text{edge}}$ *matrix of the diagonal-state clipped* M*-lattice system is lower-triangular, and one of the eigenvalues is* $a_0 + \frac{h_0}{T}$. *This can be also readily seen from (41). Since the diagonal-state clipped* M*-lattice system is totally stable by Proposition 3.1, setting* $h_0 > -Ta_0$ *guarantees that* $\vec{\chi} \in \{-1, 1\}^N$. *When* $T = 1$, *the diagonal-state clipped* M*-lattice system reduces to the reciprocal cellular neural network [2], [7], [12], [3], [16]. Thus, setting* $h_0 > -a_0$ *guarantees that* $\vec{\chi} \in \{-1, 1\}^N$, i.e., *binary outputs are obtained.*

It is interesting to study how the conditions of Proposition 4.1 vary in the limit as $T \to 0$. Intuitively, the specifications on h_0 ought to become less stringent in the high-gain limit, since any noise in the $\psi_i(t)$ (state) variables gets amplified to nearly ± 1 in the $\chi_i(t)$ (output) variables. In slightly more technical terms, a higher gain facilitates the process of making interior, face, and edge fixed points unstable. This alleviates from h_0 some of the burden of satisfying the conditions of Lemma 4.1.

Formally, multiplying throughout by T and letting $T \to 0$ in (41) gives:

$$\left\{ \begin{array}{rcl} h(\vec{0}) - \left|h(\vec{k})\right| & > & 0, \\ h(\vec{0}) & > & 0, \\ \text{where } \vec{k} & = & \arg\max_{\vec{n} \neq \vec{0}} |h(\vec{n})|. \end{array} \right. \tag{47}$$

Remark 4.3 *For the analog Hopfield network or the reciprocal cellular neural network operating in the high-gain limit, $h(\vec{0}) > 0$ guarantees that $\vec{\chi} \in \{-1, 1\}^N$. The element $h(\vec{0})$ specifies the auto-connections (*i.e.*, the amount of self-feedback of every "neuron").*

Remark 4.4 *Analysis of the general M-lattice system is also relevant for determining the stability of practical realizations of diagonal-state M-lattice systems (or analog Hopfield-type networks) [29], [30]. Since the circuit parasitics are unavoidable, the M-lattice system, (1), is a more accurate model of the actual electronic circuit than the diagonal-state M-lattice system. While the diagonal elements of **A** reflect intended circuit parameters, the off-diagonal elements capture the parasitic resistances, coupling the state nodes of the network. Therefore, prior to committing the VLSI layout of, say, an analog Hopfield network to fabrication, the parasitics have to be carefully modeled and the entire design simulated as a full M-lattice system in order to verify that the network is totally stable and produces appropriate output values.*

5 Conclusions

The M-lattice system, which evolved from reaction-diffusion, has as special cases the analog Hopfield network and the cellular neural network, and has more flexibility in how its variables interact [4], [5], [6]. We have proven the total stability of a subclass of the M-lattice system. This subclass is called the diagonal-state clipped M-lattice system and is a generalization of the cellular neural network. While the original cellular neural network is indeed totally stable, the proof appearing in [2] contains imprecise statements and omissions. In contrast, the proof presented here is more rigorous and more general, and applies to the cellular neural network as a special case.

An important problem is how to cause the outputs of various types of the M-lattice to assume binary values in equilibrium. We develop the sufficient conditions on the parameter values and study the limiting case of an infinite sigmoidal gain. The analysis concerns the stability of fixed points, making precise an earlier argument based on circuit diagrams that was published for the case of the cellular neural network [2].

For the special case of the analog Hopfield network, the analysis here explains why the output variables converge to binary values if the neuron auto-connections (*i.e.*, the diagonal elements of **A**) are non-negative. In other words, for causing binary outputs, the neuron auto-connections do not necessarily have to be strictly zero, but can also be positive. Observing this behavior in computer simulation has puzzled researchers for a period of time, and, to the best of our knowledge, this result provides the first theoretical account.

References

[1] J. J. Hopfield, “Neurons with graded response have collective computational properties like those of two-state neurons,” *Proceedings of National Academy of Sciences, USA*, vol. 81, pp. 3088–3092, May 1984.

[2] L. O. Chua and L. Yang, “Cellular neural networks: Theory,” *IEEE Transactions on Circuits and Systems*, vol. 35, pp. 1257–1272, Oct. 1988.

[3] L. O. Chua and T. Roska, “Stability of a class of nonreciprocal cellular neural networks,” *IEEE Transactions on Circuits and Systems*, vol. 37, pp. 1520–1527, Dec. 1990.

[4] A. Sherstinsky and R. W. Picard, “M-Lattice: A novel non-linear dynamical system and its application to halftoning,” in *Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing*, vol. 2, (Adelaide, Australia), pp. 565–568, Apr. 1994.

[5] A. Sherstinsky and R. W. Picard, “Restoration and enhancement of fingerprint images using M-Lattice – a novel non-linear dynamical system,” in *Proceedings of International Conference on Pattern Recognition*, (Jerusalem, Israel), Oct. 1994.

[6] A. Sherstinsky and R. W. Picard, “Orientation-sensitive image processing with M-Lattice – a novel non-linear dynamical system,” in *Proceedings of First IEEE International Conference on Image Processing*, (Austin, TX), Nov. 1994.

[7] L. O. Chua and L. Yang, “Cellular neural networks: Applications,” *IEEE Transactions on Circuits and Systems*, vol. 35, pp. 1273–1290, Oct. 1988.

[8] M. W. Hirsch, “Convergence in neural nets,” in *Proceedings of IEEE International Joint Conference on Neural Networks*, pp. II:115–125, 1987.

[9] J. Bruck and J. W. Goodman, “A generalized convergence theorem for neural networks,” *IEEE Transactions on Information Theory*, vol. 34, pp. 89–92, Sept. 1988.

[10] F. M. A. Salam, Y. Wang, and M.-R. Choi, “On the analysis of dynamic feedback neural nets,” *IEEE Transactions on Circuits and Systems*, vol. 38, pp. 196–201, Feb. 1991.

[11] M. Vidyasagar, “Minimum-seeking properties of analog neural networks with multilinear objective functions,” Tech. Rep. (Unpublished), Centre For AI & Robotics, Bangalore, India, 1993.

[12] T. Roska and L. O. Chua, “Cellular neural networks with nonlinear and delay-type template elements,” in *Proc. IEEE Int. Workshop on Cellular Neural Networks and Applications*, (Budapest, Hungary), 1990.

[13] J. J. Hopfield and D. W. Tank, ““Neural” computation of decisions in optimization problems,” *Biological Cybernetics*, vol. 52, pp. 141–152, 1985.

[14] D. Anastassiou, “Error diffusion coding for A/D conversion,” *IEEE Transactions on Circuits and Systems*, vol. 36, pp. 1175–1186, Sept. 1989.

[15] J. K. Paik and A. K. Katsaggelos, “Image restoration using a modified Hopfield network,” *IEEE Transactions on Image Processing*, vol. 1, pp. 49–63, Jan. 1992.

[16] L. O. Chua, T. Roska, *et al.*, “Special issue on cellular neural networks,” *IEEE Transactions on Circuits and Systems*, vol. 40, Mar. 1993.

[17] A. M. Turing, “The chemical basis of morphogenesis,” *Philosophical Transactions of Royal Society of London*, vol. 237, no. B, pp. 37–72, 1952.

[18] N. F. Britton, *Reaction-Diffusion Equations and Their Applications to Biology.* New York, NY: Academic Press, 1986.

[19] J. D. Murray, *Mathematical Biology.* Berlin: Springer-Verlag, 1993.

[20] A. S. Sherstinsky, *M-Lattice: A System For Signal Synthesis And Processing Based On Reaction-Diffusion.* ScD thesis, Massachusetts Institute of Technology, Cambridge, MA, 1994.

[21] W. Rudin, *Principles of Mathematical Analysis.* New York: McGraw-Hill, Inc., 1976.

[22] J. E. Marsden and A. J. Tromba, *Vector Calculus.* New York: W. H. Freeman and Company, 1981.

[23] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations And Boundary Value Problems.* New York: Wiley, 1977.

[24] M. Vidyasagar, *Nonlinear Systems Analysis.* New Jersey: Prentice-Hall, 1978.

[25] G. Strang, *Linear Algebra and Its Applications.* Academic Press, 1980.

[26] G. Strang, *Introduction to Applied Mathematics.* Cambridge, MA: Wellesley-Cambridge Press, 1986.

[27] J. H. Wilkinson, *The Algebraic Eigenvalue Problem.* Oxford: Clarendon Press, 1965.

[28] G. H. Golub and C. F. V. Loan, *Matrix Computations.* Baltimore, Maryland: The Johns Hopkins University Press, 1983.

[29] C. Mead, *Analog VLSI and Neural Systems.* Addison-Wesley, 1989.

[30] B. W. Lee and B. J. Sheu, *Hardware Annealing in Analog VLSI Neurocomputing.* Kluwer Academic Publishers, 1991.